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Letter to the Editor

On the eigencharacteristics of multi-step beams carrying a tip mass subjected to non-homogeneous external viscous damping

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1. Introduction

An interesting study [1] dealt with the determination of the eigencharacteristics of a continuous beam model with damping by using the separation of variables approach. The beam considered had different stiffness, damping and mass properties in each of its two parts. Motivated by this study, another publication [2] was concerned with an axially vibrating rod consisting of two parts as a counterpart of the previous publication. Unlike [1], where overdamped and underdamped "modes" were investigated separately, the two modes were handled simultaneously in Ref. [2], again via separation of variables approach. In Ref. [3], the present authors extended their results in Ref. [2] to axially vibrating multi-step rods. They also gave a second method for the determination of the eigencharacteristics, which is principally a transfer matrix method. The present work is concerned essentially with a mechanical system, similar to that investigated in Ref. [3], but here bending vibrations of a multi-step Bernoulli–Euler beam are investigated as a counterpart of that publication.

2. Theory

Let us assume that a laterally vibrating beam carrying a tip mass M consists of n parts, the ith of which has length L_i, bending rigidity E_iI_i, external viscous damping coefficient c_i, and mass per unit length m_i, respectively. These parameters are assumed to be constant along each beam segment, and contain contributions from the beam and any surrounding medium. Fig. 1 shows the beam diagrammatically.

Due to the presence of external viscous damping it is more appropriate to work with complex variables. It will be assumed that the bending displacements $w_i(x, t)$, $(i = 1, \dots, n)$ of several parts of the beam are the real parts of some complex quantities denoted as $z_i(x, t)$. Keeping in mind that

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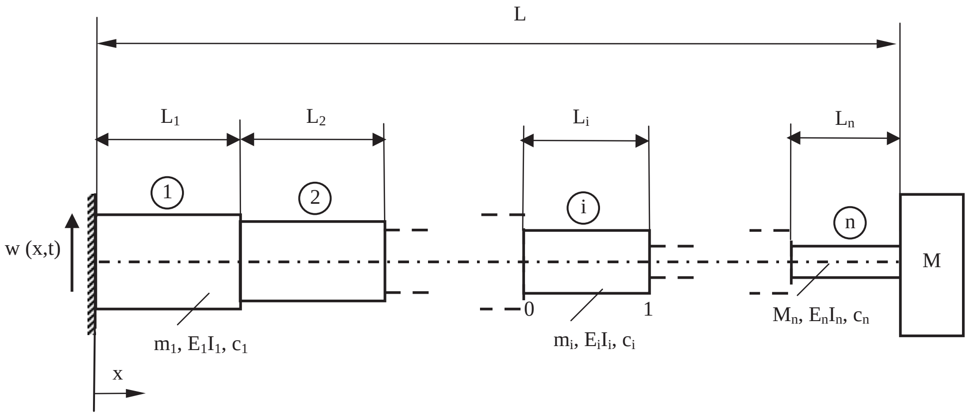

Fig. 1. Laterally vibrating elastic beam having several parts and carrying a tip mass.

actually the authors are interested only in the real parts of the expressions below, the equations of motion of the beam can be written as

$$k_i z_i^{iv}(x,t) + m_i \ddot{z}_i(x,t) + c_i \dot{z}_i(x,t) = 0 \quad (i = 1, \ldots, n) \tag{1}$$

with $k_i = E_i I_i$, where x is the axial position along the beam. Dots and primes denote partial derivatives with respect to time t and position co-ordinate x.

The corresponding boundary conditions are

$$\begin{aligned} &z_1(0,t) = 0,\;\; z_1'(0,t) = 0,\;\; z_{i-1}(\bar{L}_i,t) = z_i(\bar{L}_i,t),\;\; z_{i-1}'(\bar{L}_i,t) = z_i'(\bar{L}_i,t), \\ &k_{i-1} z_{i-1}''(\bar{L}_i,t) = k_i z_i''(\bar{L}_i,t),\;\; k_{i-1} z_{i-1}'''(\bar{L}_i,t) = k_i z_i'''(\bar{L}_i,t), \quad (i = 2, \ldots, n) \\ &k_n z_n''(\bar{L}_{n+1},t) = 0,\;\; k_n z_n'''(\bar{L}_{n+1},t) - M \ddot{z}_n(\bar{L}_{n+1},t) = 0, \end{aligned} \tag{2}$$

where

$$\bar{L}_r = \sum_{j=1}^{r-1} L_j \quad (r = 2, \ldots, n+1). \tag{3}$$

Let us assume that

$$z_i(x,t) = Z_i(x) D_i(t) \quad (i = 1, \ldots, n) \tag{4}$$

according to the separation of the variables approach, where both functions $Z_i(x)$ and $D_i(t)$ are complex functions in general. Substituting (4) into Eq. (1) gives

$$\frac{k_i}{m_i} \frac{Z_i^{iv}(x)}{Z_i(x)} = -\frac{\ddot{D}_i(t) + \left(\dfrac{c_i}{m_i}\right) \dot{D}_i(t)}{D_i(t)} := \kappa_i, \tag{5}$$

where the κ_i are complex constants to be determined. Here, primes and dots denote derivatives with respect to position x and time t. To satisfy the boundary conditions (2) the corresponding time functions must be equal, so that $D_i(t) = D(t)$, $(i = 1, \ldots, n)$. Thus, the differential equations for $Z_i(x)$ may be written using Eq. (5) as follows:

$$Z_i^{iv}(x) - \frac{m_i}{k_i} \kappa_i Z_i(x) = 0 \quad (i = 1, \ldots, n). \tag{6}$$

The time function is assumed now as an exponential function

$$D(t) = \mathrm{e}^{\lambda t}, \tag{7}$$

where λ represents an eigenvalue of the system which is complex in general. With this $D(t)$, the second equality in Eq. (5) gives

$$\kappa_i = -\frac{c_i}{m_i}\lambda - \lambda^2 \quad (i = 1, \ldots, n). \tag{8}$$

With the abbreviation

$$v_i^4 = \frac{m_i}{k_i}\kappa_i \quad (i = 1, \ldots, n) \tag{9}$$

the first equation in Eq. (5) can be written as

$$Z_i^{iv}(x) - v_i^4 Z_i(x) = 0 \quad (i = 1, \ldots, n). \tag{10}$$

The general solutions of the differential equation (10) can be expressed as

$$Z_i(x) = \bar{A}_i \mathrm{e}^{v_i x} + \bar{B}_i \mathrm{e}^{-v_i x} + \bar{C}_i \mathrm{e}^{\mathrm{j}v_i x} + \bar{D}_i \mathrm{e}^{-\mathrm{j}v_i x} \quad (i = 1, \ldots, n), \tag{11}$$

where $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ and $\bar{D}_i$ denote complex constants to be determined and $\mathrm{j} = \sqrt{-1}$. In terms of the $Z_i(x)$, and with the aid of Eq. (7) the boundary conditions in Eq. (2) can be formulated as

$$\begin{aligned}
&Z_1(0) = 0,\ \ Z_1'(0) = 0,\ \ Z_{i-1}(\bar{L}_i) = Z_i(\bar{L}_i),\ \ Z_{i-1}'(\bar{L}_i) = Z_i'(\bar{L}_i),\\
&k_{i-1}Z_{i-1}''(\bar{L}_i) = k_i Z_i''(\bar{L}_i),\ \ k_{i-1}Z_{i-1}'''(\bar{L}_i) = k_i Z_i'''(\bar{L}_i), \quad (i = 2, \ldots, n),\\
&k_n Z_n''(\bar{L}_{n+1}) = 0,\ \ k_n Z_n'''(\bar{L}_{n+1}) - M\lambda^2 Z_n(\bar{L}_{n+1}) = 0.
\end{aligned} \tag{12}$$

The substituting expressions (11) into (12) yields the following set of $4n$ homogeneous equations for the determination of the $4n$ unknowns: $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ and $\bar{D}_i$ $(i = 1, \ldots, n)$:

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0\\
1 & -1 & \mathrm{j} & -\mathrm{j} & 0 & 0 & \cdots & 0 & 0\\
\mathrm{e}^{v_1\bar{L}_2} & \mathrm{e}^{-v_1\bar{L}_2} & \mathrm{e}^{\mathrm{j}v_1\bar{L}_2} & \mathrm{e}^{-\mathrm{j}v_1\bar{L}_2} & \text{-}\mathrm{e}^{v_2\bar{L}_2} & \mathrm{e}^{-v_2\bar{L}_2} & \cdots & 0 & 0\\
v_1\mathrm{e}^{v_1\bar{L}_2} & -v_1\mathrm{e}^{-v_1\bar{L}_2} & \mathrm{j}v_1\mathrm{e}^{\mathrm{j}v_1\bar{L}_2} & -\mathrm{j}v_1\mathrm{e}^{-\mathrm{j}v_1\bar{L}_2} & -v_2\mathrm{e}^{v_2\bar{L}_2} & v_2\mathrm{e}^{-v_2\bar{L}_2} & \cdots & 0 & 0\\
k_1v_1^2\mathrm{e}^{v_1\bar{L}_2} & k_1v_1^2\mathrm{e}^{-v_1\bar{L}_2} & -k_1v_1^2\mathrm{e}^{\mathrm{j}v_1\bar{L}_2} & -k_1v_1^2\mathrm{e}^{-\mathrm{j}v_1\bar{L}_2} & -k_2v_2^2\mathrm{e}^{v_2\bar{L}_2} & -k_2v_2^2\mathrm{e}^{-v_2\bar{L}_2} & \cdots & 0 & 0\\
k_1v_1^3\mathrm{e}^{v_1\bar{L}_2} & -k_1v_1^3\mathrm{e}^{-v_1\bar{L}_2} & -\mathrm{j}k_1v_1^3\mathrm{e}^{\mathrm{j}v_1\bar{L}_2} & \mathrm{j}k_1v_1^3\mathrm{e}^{-\mathrm{j}v_1\bar{L}_2} & -k_2v_2^3\mathrm{e}^{v_2\bar{L}_2} & k_2v_2^3\mathrm{e}^{-v_2\bar{L}_2} & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & \mathrm{e}^{v_2\bar{L}_3} & \mathrm{e}^{-v_2\bar{L}_3} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & k_nv_n^2\mathrm{e}^{\mathrm{j}v_n\bar{L}_{n+1}} & k_nv_n^2\mathrm{e}^{-\mathrm{j}v_n\bar{L}_{n+1}}\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -(\mathrm{j}k_nv_n^3 + M\lambda^2)\,\mathrm{e}^{\mathrm{j}v_n\bar{L}_{n+1}} & (\mathrm{j}k_nv_n^3 - M\lambda^2)\,\mathrm{e}^{-\mathrm{j}v_n\bar{L}_{n+1}}
\end{bmatrix}
\begin{bmatrix}\bar{A}_1\\ \bar{B}_1\\ \bar{C}_1\\ \bar{D}_1\\ \bar{A}_2\\ \bar{B}_2\\ \bar{C}_2\\ \vdots\\ \bar{C}_n\\ \bar{D}_n\end{bmatrix} = 0. \tag{13}$$

Let the $(4n \times 4n)$ matrix of the coefficients in Eq. (13) be denoted by $\mathbf{A}$. For a non-trivial solution, the determinant of this matrix should be zero:

$$\det \mathbf{A}(v_1, \ldots, v_n) = 0. \tag{14}$$

If in Eq. (14) the tip mass M approaches infinity, then the characteristic equation of the bending vibrations of a beam clamped at both ends is obtained. Equating to zero the tip mass M yields the characteristic equation of the clamped–free beam.

Using the definitions given by Eqs. (8) and (9) v_i's, $(i = 1, \ldots, n)$ can be expressed as functions of the eigenvalue λ:

$$v_i(\lambda) = \pm\left\{\frac{m_i}{k_i}\left[\left(\frac{c_i}{m_i}\right)\lambda + \lambda^2\right]\right\}^{1/4} \quad (i = 1, \ldots, n). \tag{15}$$

Hence Eq. (14) becomes

$$\det \mathbf{A}(v_1(\lambda), \ldots, v_n(\lambda)) = \det \mathbf{A}(\lambda) = 0 \tag{16}$$

from which λ can be obtained, which is a complex number in general. Now via (15) v_i's can be obtained. If these are substituted into the coefficients matrix $\mathbf{A}$ in Eq. (13), the unknowns $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ and $\bar{D}_i$, $(i = 1, \ldots, n)$ can be determined up to an arbitrary constant. Hence, $Z_i(x)$, $(i = 1, \ldots, n)$ in Eq. (11) are obtained.

Let us return to Eq. (4) considering expression (11).

Introducing

$$\begin{aligned}&\lambda = \lambda_{\mathrm{re}} + \mathrm{j}\lambda_{\mathrm{im}}, \quad v_i = v_{i\,\mathrm{re}} + \mathrm{j}v_{i\,\mathrm{im}}, \quad \bar{A}_i = \bar{A}_{i\,\mathrm{re}} + \mathrm{j}\bar{A}_{i\,\mathrm{im}},\\ &\bar{B}_i = \bar{B}_{i\,\mathrm{re}} + \mathrm{j}\bar{B}_{i\,\mathrm{im}}, \quad \bar{C}_i = \bar{C}_{i\,\mathrm{re}} + \mathrm{j}\bar{C}_{i\,\mathrm{im}}, \quad \bar{D}_i = \bar{D}_{i\,\mathrm{re}} + \mathrm{j}\bar{D}_{i\,\mathrm{im}},\end{aligned} \tag{17}$$

the bending displacements of the n beam portions $w_i(x,t)$ are determined, after lengthy calculations as

$$w_i(x,t) = \mathrm{Re}[z_i(x,t)] = \mathrm{e}^{\lambda_{\mathrm{re}}t}C_{i,1}(x)\cos\lambda_{\mathrm{im}}t + \mathrm{e}^{\lambda_{\mathrm{re}}t}C_{i,2}(x)\sin\lambda_{\mathrm{im}}t, \tag{18}$$

where the following abbreviations are introduced:

$$\begin{aligned}C_{i,1}(x) = {}& \mathrm{e}^{v_{i\,\mathrm{re}}x}(\bar{A}_{i\,\mathrm{re}}\cos v_{i\,\mathrm{im}}x - \bar{A}_{i\,\mathrm{im}}\sin v_{i\,\mathrm{im}}x) + \mathrm{e}^{-v_{i\,\mathrm{re}}x}(\bar{B}_{i\,\mathrm{re}}\cos v_{i\,\mathrm{im}}x + \bar{B}_{i\,\mathrm{im}}\sin v_{i\,\mathrm{im}}x)\\ &+ \mathrm{e}^{-v_{i\,\mathrm{im}}x}(\bar{C}_{i\,\mathrm{re}}\cos v_{i\,\mathrm{re}}x - \bar{C}_{i\,\mathrm{im}}\sin v_{i\,\mathrm{re}}x) + \mathrm{e}^{v_{i\,\mathrm{im}}x}(\bar{D}_{i\,\mathrm{re}}\cos v_{i\,\mathrm{re}}x + \bar{D}_{i\,\mathrm{im}}\sin v_{i\,\mathrm{re}}x),\\ C_{i,2}(x) = {}& -\mathrm{e}^{v_{i\,\mathrm{re}}x}(\bar{A}_{i\,\mathrm{re}}\sin v_{i\,\mathrm{im}}x + \bar{A}_{i\,\mathrm{im}}\cos v_{i\,\mathrm{im}}x) + \mathrm{e}^{-v_{i\,\mathrm{re}}x}(\bar{B}_{i\,\mathrm{re}}\sin v_{i\,\mathrm{im}}x - \bar{B}_{i\,\mathrm{im}}\cos v_{i\,\mathrm{im}}x)\\ &- \mathrm{e}^{-v_{i\,\mathrm{im}}x}(\bar{C}_{i\,\mathrm{re}}\sin v_{i\,\mathrm{re}}x + \bar{C}_{i\,\mathrm{im}}\cos v_{i\,\mathrm{re}}x) + \mathrm{e}^{v_{i\,\mathrm{im}}x}(\bar{D}_{i\,\mathrm{re}}\sin v_{i\,\mathrm{re}}x - \bar{D}_{i\,\mathrm{im}}\cos v_{i\,\mathrm{re}}x).\end{aligned} \tag{19}$$

The expressions of the bending displacements can be put in a more compact form, as

$$w_i(x,t) = \mathrm{e}^{\lambda_{\mathrm{re}}t}C_i(x)\cos(\lambda_{\mathrm{im}}t - \varepsilon_i(x)) \tag{20}$$

with

$$\begin{aligned}C_i(x) &= \sqrt{C_{i,1}^2(x) + C_{i,2}^2(x)},\\ \tan\varepsilon_i(x) &= \frac{C_{i,2}(x)}{C_{i,1}(x)}.\end{aligned} \tag{21}$$

$w_i(x,t)$, $(i = 1, \ldots, n)$ determine the bending displacement distribution over the length of the viscously damped beam when it vibrates at an eigenvalue λ. Due to the apparent presence of a phase which is a function of the position co-ordinate x, the authors used the expressions "mode" or "eigenfunction" unwillingly, unlike in Ref. [1]. Whenever necessary, those words were used in quotation marks. $C_i(x)$ which is simply the absolute value of $Z_i(x)$, i.e., $\mathrm{Abs}(Z_i(x))$, represents the amplitude distribution over the ith step of the beam.

With the given method above, the eigenvalues λ of the laterally vibrating multi-step beam are found as the roots of a complex determinant of size $4n \times 4n$ if the beam under consideration consists of n portions (steps). In the following, an alternative form of the characteristic equation will be given which could be preferable for numerical calculations especially for large n values. The second form is essentially a transfer matrix method, which also makes use of the separation of variables approach at the beginning. Li and co-workers applied it in a series of papers successfully to the longitudinal vibrations of rods and rod systems with variable cross-sections [4–6].

Here, the cross-sections of the beam are assumed to be constant along a beam portion, but external viscous damping is allowed to act on the beam.

The bending displacement distribution along the ith portion of the beam given in Eq. (11) can be rewritten as

$$Z_i(x) = C_{i,1}S_{i,1}(x) + C_{i,2}S_{i,2}(x) + C_{i,3}S_{i,3}(x) + C_{i,4}S_{i,4}(x) \quad (i = 1, \ldots, n), \tag{22}$$

where the abbreviations

$$S_{i,1}(x) = \mathrm{e}^{\nu_i x},\;\; S_{i,2}(x) = \mathrm{e}^{-\nu_i x},\;\; S_{i,3}(x) = \mathrm{e}^{\,\mathrm{j}\nu_i x},\;\; S_{i,4}(x) = \mathrm{e}^{-\mathrm{j}\nu_i x} \tag{23}$$

are introduced. Let us assume in what follows, that $x = 0$ corresponds to the left end of the ith beam portion. The results in Ref. [4] can be adopted appropriately.

The relationship between the parameters, $Z_{i,1}$ (bending displacement), $Z'_{i,1}$ (slope), $M_{i,1}$ (bending moment) and $Q_{i,1}$ (shear force) at the right end (denoted by subscript 1 in Fig. 1) and at the left end of the ith portion (subscript 0 in Fig. 1) can be expressed in matrix notations as

$$\begin{bmatrix} Z_{i,1} \\ Z'_{i,1} \\ M_{i,1} \\ Q_{i,1} \end{bmatrix} = \mathbf{T}_i \begin{bmatrix} Z_{i,0} \\ Z'_{i,0} \\ M_{i,0} \\ Q_{i,0} \end{bmatrix} \tag{24}$$

in which

$$\mathbf{T}_i = \begin{bmatrix} S_{i,1}(L_i) & S_{i,2}(L_i) & S_{i,3}(L_i) & S_{i,4}(L_i) \\ S'_{i,1}(L_i) & S'_{i,2}(L_i) & S'_{i,3}(L_i) & S'_{i,4}(L_i) \\ k_iS''_{i,1}(L_i) & k_iS''_{i,2}(L_i) & k_iS''_{i,3}(L_i) & k_iS''_{i,4}(L_i) \\ -k_iS'''_{i,1}(L_i) & -k_iS'''_{i,2}(L_i) & -k_iS'''_{i,3}(L_i) & -k_iS'''_{i,4}(L_i) \end{bmatrix}$$
$$\times \begin{bmatrix} S_{i,1}(0) & S_{i,2}(0) & S_{i,3}(0) & S_{i,4}(0) \\ S'_{i,1}(0) & S'_{i,2}(0) & S'_{i,3}(0) & S'_{i,4}(0) \\ k_iS''_{i,1}(0) & k_iS''_{i,2}(0) & k_iS''_{i,3}(0) & k_iS''_{i,4}(0) \\ -k_iS'''_{i,1}(0) & -k_iS'''_{i,2}(0) & -k_iS'''_{i,3}(0) & -k_iS'''_{i,4}(0) \end{bmatrix}^{-1} \quad (i = 1, \ldots, n), \tag{25}$$

where a prime denotes derivative with respect to x. The matrix $\mathbf{T}_i$ is called the transfer matrix because it transfers the parameters at the end 0 to those at the end 1 of the ith step beam. It can be shown that the transfer matrix transferring the parameters at the station 0 of the first beam step to the right end of the multi-step beam carrying the tip mass is

$$\mathbf{T} = \mathbf{T}_M \cdot \mathbf{T}_n \ldots \mathbf{T}_1 = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix}, \tag{26}$$

where the matrix

$$\mathbf{T}_M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda^2 M & 0 & 0 & 1 \end{bmatrix}, \tag{27}$$

accounts for the tip mass.

In the case of the system in Fig. 1, the boundary conditions are such that the bending displacement and slope at the left end and the bending moment and shear force at the right end of the multi-step beam system vanish. This leads to the characteristic equation:

$$\begin{vmatrix} T_{33}(v_1(\lambda)\ldots, v_n(\lambda)) & T_{34}(v_1(\lambda)\ldots, v_n(\lambda)) \\ T_{43}(v_1(\lambda)\ldots, v_n(\lambda)) & T_{44}(v_1(\lambda)\ldots, v_n(\lambda)) \end{vmatrix} = \begin{vmatrix} T_{33}(\lambda) & T_{34}(\lambda) \\ T_{43}(\lambda) & T_{44}(\lambda) \end{vmatrix} = 0. \tag{28}$$

If there is no tip mass, the matrix $\mathbf{T}_M$ reduces to the 4×4 unit matrix. Hence, the overall transfer matrix $\mathbf{T}$ reduces to the product of the n sub-transfer matrices $\mathbf{T}_i$:

$$\mathbf{T} = \mathbf{T}_n \ldots \mathbf{T}_1. \tag{29}$$

The characteristic equation (28) holds formally where T_{33}, T_{34}, T_{43} and T_{44} denote in this case the corresponding elements of the matrix $\mathbf{T}$ given in Eq. (29). In the case when the tip mass tends to infinity, i.e., the beam is clamped–clamped, the boundary conditions to be impose are that the bending displacements and slopes at both ends of the multi-step beam vanish which leads to the characteristic equation

$$\begin{vmatrix} T_{13}(\lambda) & T_{14}(\lambda) \\ T_{23}(\lambda) & T_{24}(\lambda) \end{vmatrix} = 0, \tag{30}$$

where the overall transfer matrix $\mathbf{T}$ is given by Eq. (29).

3. Numerical evaluations

This section is devoted to the numerical evaluation of the expressions obtained above. The computation will be demonstrated using a 3 step beam with the parameters given in Table 1.

Three cases are considered: In the first case, it is assumed that there is no tip mass, i.e., $M = 0$ (case I), in the second case, there is a tip mass: $M \neq 0$ (case II) and finally, the tip mass is infinite, i.e., the 3 step beam is clamped–clamped (case III). As can be seen from Table 1, in all cases, the physical parameters of the beam steps are equal. The tip mass in case II is $M = 50\,\text{kg}$.

Table 1
Physical parameters of the beams in cases I, II and III

	Step 1	Step 2	Step 3
L_i (m)	1	2	2
M_i (kg/m)	20	10	10
C_i (kg/ms)	0	0	1000
E_iI_i (N m^2)	200	100	50

Table 2
"Lower" eigenvalues for case I

From Eq. (16)	From Eq. (29)
−0.00300	−0.00300
−0.15164	−0.15164
−2.92814	−2.92814
−0.73194 ± 6.02561i	−0.73194 ± 6.02561i
−1.78680 ± 16.57028i	−1.78680 ± 16.57028i
−3.75927 ± 34.38621i	−3.75927 ± 34.38621i

Table 3
"Lower" eigenvalues for case II

From Eq. (16)	From Eq. (28)
−0.00300	−0.00300
−0.15346	−0.15346
−14.81754	−14.81754
−0.70181 ± 6.03149i	−0.70181 ± 6.03149i
−1.81269 ± 16.58422i	−1.81269 ± 16.58422i
−3.76469 ± 34.35086i	−3.76469 ± 34.35086i

Table 2 gives the "first" six eigenvalues of the system for case I. It is seen that the physical parameters lead to both overdamped and underdamped "modes". The numerical values in the first column represent the results of finding the roots of the determinantal equation in Eq. (16) whereas those of the second column are results of Eq. (29) based on the transfer matrix method. The first columns of Tables 3 and 4 are based on Eq. (16), whereas those of the second columns represent the roots of Eqs. (28) and (30), respectively. The numbers in both columns are exactly the same. The upper part of Fig. 2 shows for case I the three-dimensional plots of $w_i(x,t)$ for the first three overdamped eigenvalues. In the lower part, the amplitude distribution Abs($Z_i(x)$) is plotted which is composed of 3 curves of Abs($Z_i(x)$) corresponding to $n = 3$ steps on the beam.

As in Fig. 2, the upper part of Fig. 3 shows for case I the $w_i(x,t)$-surfaces for the "first" three underdamped eigenvalues. The lower part gives the amplitude distributions.

Table 3 gives the "first" six eigenvalues for the case II. Again, there are both those of the overdamped and the underdamped "modes". The numerical values in both columns are the same.

Table 4
"Lower" eigenvalues for case III

From Eq. (16)	From Eq. (30)
−0.16489	−0.16489
−2.91342	−2.91342
−36.00952	−36.00952
-0.73285 ± 6.02476i	-0.73285 ± 6.02476i
-1.78696 ± 16.57041i	-1.78696 ± 16.57041i
-3.79523 ± 34.38624i	-3.79523 ± 34.38624i

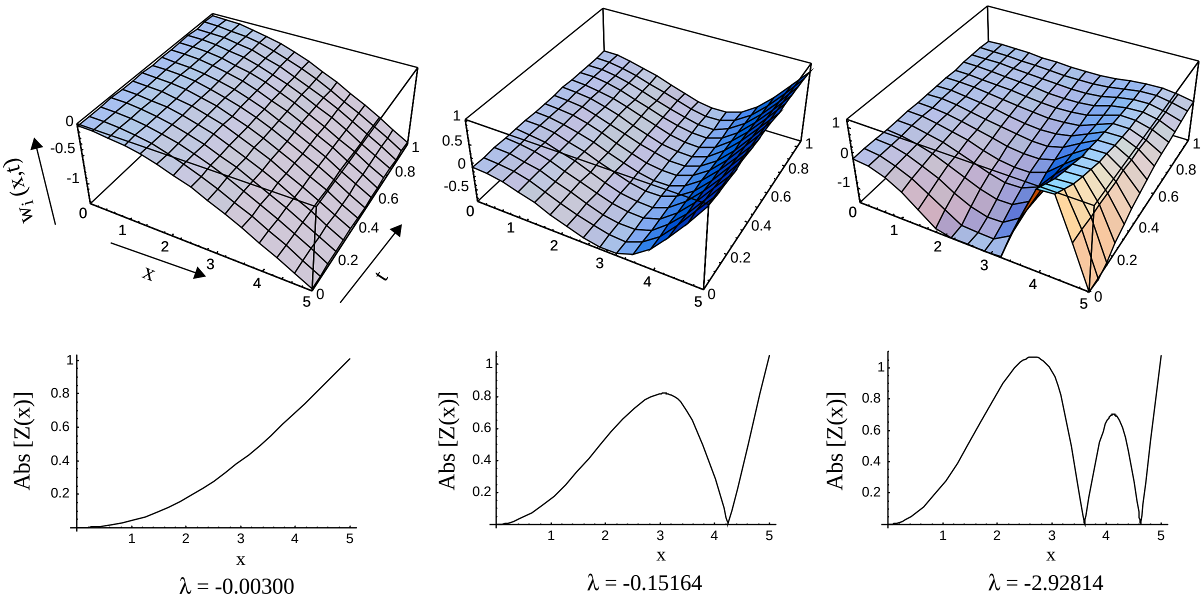

Fig. 2. Three-dimensional plots of $w_i(x,t)$-surfaces and amplitude distributions Abs($Z(x)$), corresponding to the first three overdamped eigenvalues for case I.

Figs. 4 and 5 are concerned with case II. As in Figs. 2 and 3, the upper parts of Figs. 4 and 5 reflect the $w_i(x,t)$ plots of the "first" three overdamped and underdamped eigenvalues, respectively. The lower parts give again the corresponding amplitude distributions.

Finally, Table 4 collects the "first" six eigenvalues for case III, where the numerical values in both columns are the same, as previously. Figs. 6 and 7 are concerned with this case. As above, the upper parts of these figures give the $w_i(x,t)$ plots of the "first" three overdamped and underdamped eigenvalues, respectively. The lower parts give the corresponding amplitude distributions.

Comparison of the overdamped eigenvalues in Tables 2 and 3 reveals that the attachment of the tip mass causes the eigenvalues to be greater in absolute values which increases the damping effect. In comparison to Table 3 (case II), the overdamped eigenvalues in Table 4 (case III) have greater absolute values. In other words, the "eigenmotions" of the clamped–clamped beam corresponding

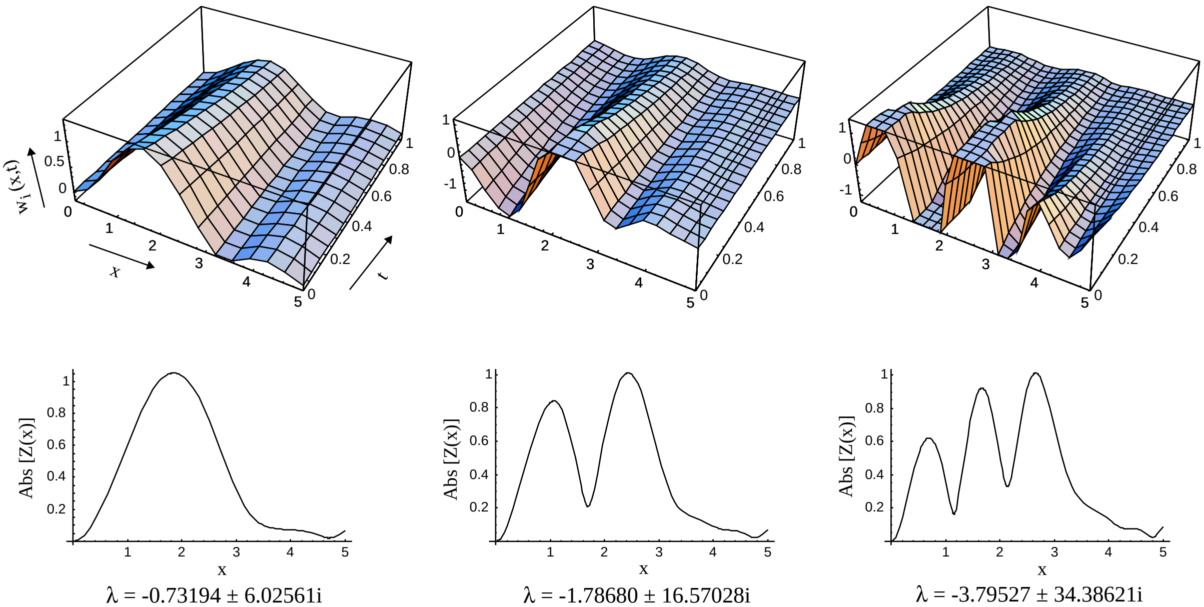

Fig. 3. Three-dimensional plots of $w_i(x,t)$-surfaces and amplitude distributions Abs($Z(x)$), corresponding to the "first" three underdamped eigenvalues for case I.

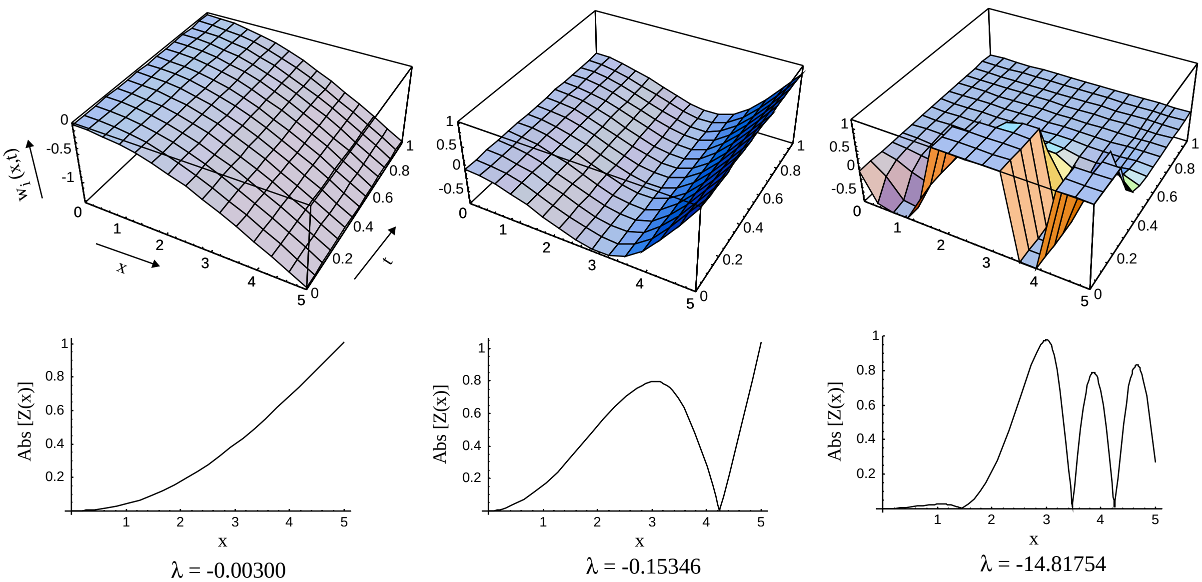

Fig. 4. Three-dimensional plots of $w_i(x,t)$-surfaces and amplitude distributions Abs($Z(x)$), corresponding to the first three overdamped eigenvalues for case II.

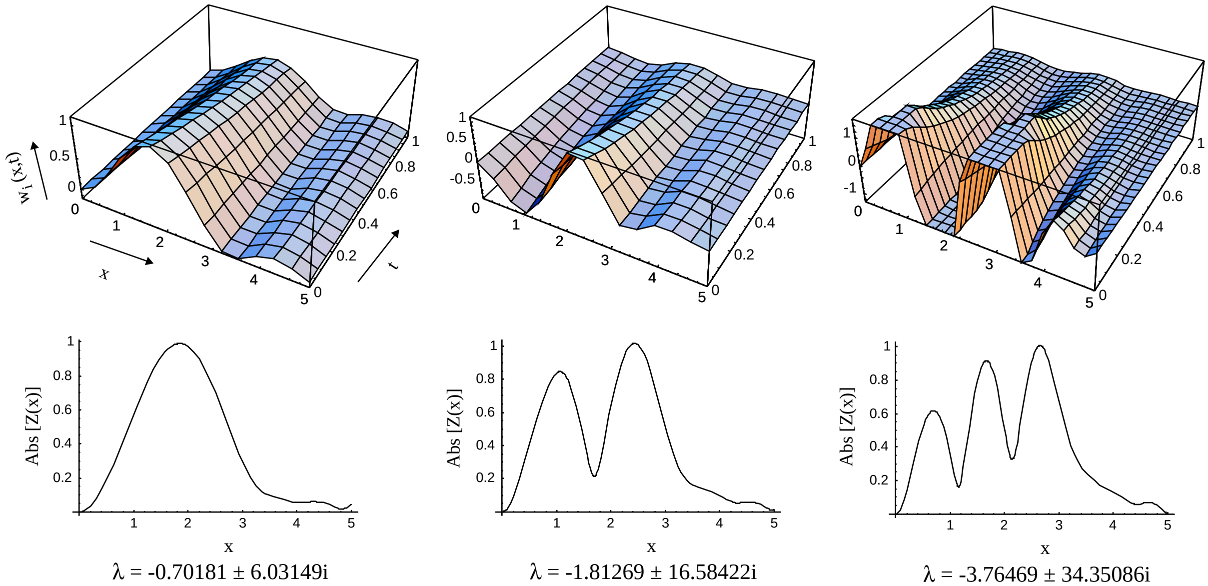

Fig. 5. Three-dimensional plots of $w_i(x, t)$-surfaces and amplitude distributions Abs($Z(x)$), corresponding to the "first" three underdamped eigenvalues for case II.

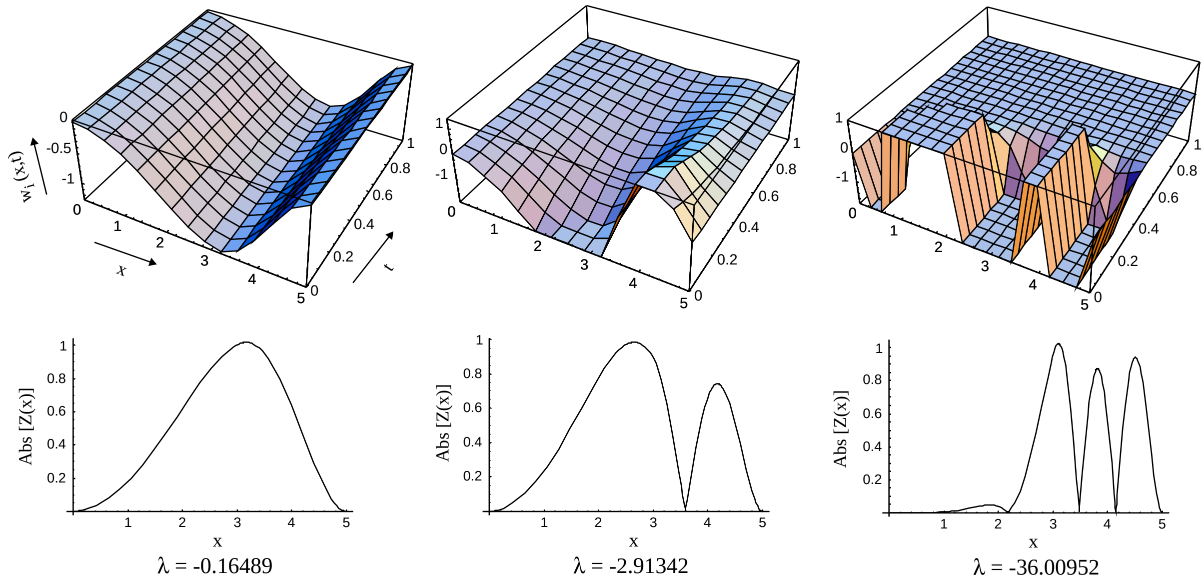

Fig. 6. Three-dimensional plots of $w_i(x, t)$-surfaces and amplitude distributions Abs($Z(x)$), corresponding to the first three overdamped eigenvalues for case III.

to these eigenvalues damp out more rapidly than in case of the beam carrying a tip mass. It is worth noting that the conclusions drawn may not be general and they are valid for the examples considered, because the results depend greatly on the geometry of the system and where the damped portions are located.

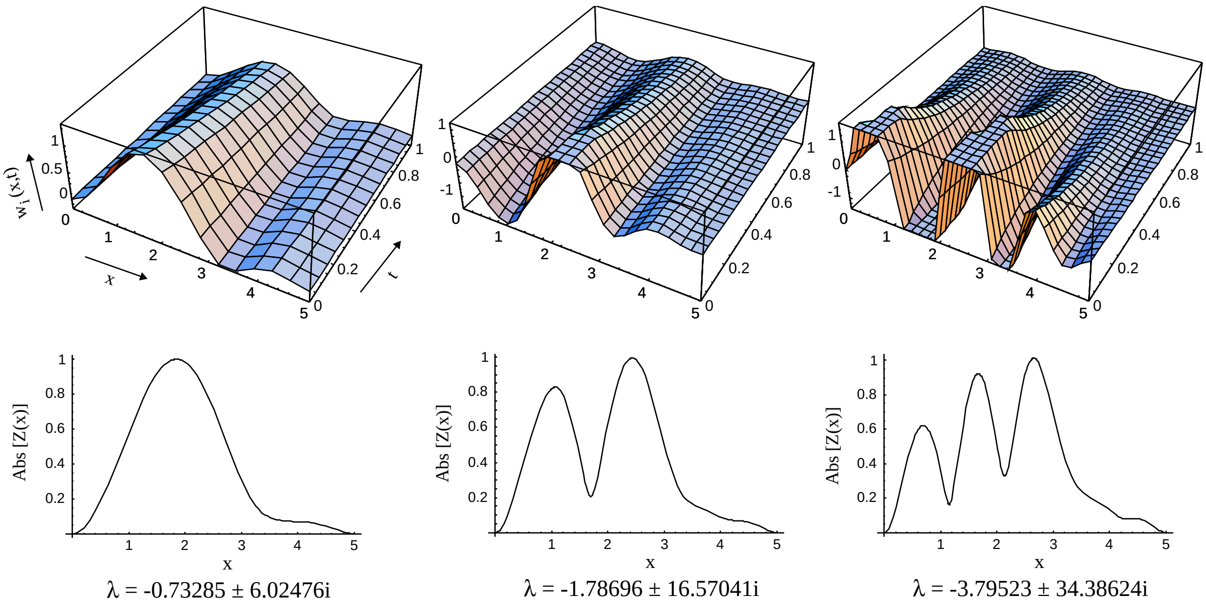

Fig. 7. Three-dimensional plots of $w_i(x, t)$-surfaces and amplitude distributions Abs($Z(x)$), corresponding to the "first" three underdamped eigenvalues for case III.

Concerning the number of "nodes", the behaviours of the three cases are similar for overdamped "modes". Within the numerical precision of the 10^{-7}, the points with very small amplitudes are denoted practically as "nodes". As can be seen from the lower parts of Figs. 2, 4 and 6, the second and third "modes" reveal 1 "node", whereas the third "modes" of cases I, II and III reveal, 2, 3 and 3 nodes, respectively. Further, the nodes are located in the damped portion of the beam. On the contrary, underdamped "modes", reveal no "nodes" as can be seen from Figs. 3, 5 and 7.

In accordance with the fact that the third portion of the beam is damped and the first two are not, one observes that the majority of the displacements in the "lower" underdamped "modes" is local to the undamped, namely to the first two parts of the beam.

4. Conclusions

This study is concerned with the establishment of two methods for computing the eigencharacteristics of a continuous beam, carrying a tip mass, consisting of several parts having different physical parameters and subjected to external viscous damping. Both methods use separation of variables approach at the beginning and differ, actually, in the solution of the corresponding ordinary differential equation. The second method is referred to as the transfer matrix method in the literature. Excellent agreement of the numerical results for three sample systems obtained via the two methods justifies the reliability of the formulae established.

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